A family of exponents from a fractal model of viscous fingering and DLA

1987 J. Phys. A: Math. Gen. 20 5961

(http://iopscience.iop.org/0305-4470/20/17/029)

J. Phys. A: Math. Gen. 20 (1987) 5961-5969. Printed in the UK

A family of exponents from a fractal model of viscous fingering and DLA

M J Blunt and R C Ball

Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CBE 0HE, UK

Received 2 March 1987, in final form 22 June 1987

Abstract. We present a simple fractal model of the interface between oil and water in viscous fingering or the cluster boundary in diffusion-limited aggregation (DLA). The continuum equations for immiscible displacement of Newtonian fluids with no surface tension in a porous medium are solved analytically to find the flow velocity on the boundary. The moments of the velocity distribution scale with the size of the system yielding a family of exponents. The scaling depends on the geometry of the interface and the oil/water viscosity ratio. For an infinite viscosity ratio, which corresponds to DLA, the form of the scaling is in qualitative agreement with the numerical results of Amitrano *et al.* The method can be easily extended to study other systems with a fractal boundary condition.

For an incompressible Newtonian fluid in a porous medium obeying Darcy's law,

$$V = (k/\eta)\nabla\phi \tag{1}$$

$$\nabla \cdot V = 0 \tag{2}$$

where V is the fluid velocity, k the porosity, ϕ the pressure and η the viscosity.

The pressure obeys Laplace's equation, $\nabla^2\phi = 0$. For immiscible displacement with negligible surface tension, the pressure and normal fluid velocity are continuous at the boundary between the two fluids.

For diffusion-limited aggregation (DLA) (Witten and Sander 1981) the concentration c of walkers outside the cluster obeys the steady-state diffusion equation, $\nabla^2 c = 0$, with $c = 0$ on the cluster and a constant at infinity. The average growth speed V on the perimeter is given by the incident flux of particles

$$V = \nabla c \cdot \boldsymbol{n} \tag{3}$$

where $\boldsymbol{n}$ is a unit normal to the cluster.

As pointed out by Paterson (1984) these are the same equations as for viscous fingering where the displaced fluid has an infinite viscosity. In such a case DLA-type patterns have been seen experimentally for fingering in random porous networks (Chen and Wilkinson 1985, Nittmann *et al* 1985, 1986, Maloy *et al* 1985, Ben-Jacob *et al* 1985, 1986, Daccord *et al* 1986, Van Damme *et al* 1986, Nittmann 1986, Buka *et al* 1986). Also other systems obeying similar continuum equations but with different microscopic mechanisms produce closely related patterns, such as electrodeposition (Brady and Ball 1984, Grier *et al* 1986, Sawada *et al* 1986), dielectric breakdown of an insulator in a high electric field (Niemeyer *et al* 1984), dendritic solidification (Langer 1980), diffusion-limited polymerisation (Kaufman *et al* 1986) and single crystalline domains in lipid monolayers (Miller *et al* 1986). In all these examples the boundary on which growth occurs has structure of many sizes and can be characterised by a fractal dimension, d_f (Mandelbrot 1982). Moreover, there are other situations where we want to understand the behaviour of a Laplacian field near a static rigid fractal (Cates and Witten 1986). It could arise that catalysis on a fractal substrate is limited by the steady-state diffusion of free reactants onto the adsorbing surface. The flow velocity of a fluid near a fractal in solution also obeys Laplace's equation, although here we have a vector rather than a scalar field.

For DLA we define a probability p_i that site i on the perimeter Γ next becomes part of the cluster. Then we look at the moments:

$$Z(q)=\sum_{i\in\Gamma} p_i^q \sim L^{-(q-1)D(q)} \tag{4}$$

in which $(q-1)D(q)$ was originally introduced to describe turbulence (Mandelbrot 1974) and later to characterise random resistor networks (de Arcangelis *et al* 1985). This simple description reveals a rich scaling structure for DLA (Halsey *et al* 1986b). An extension of this idea is to look at the moments of the velocity V in viscous fingering and how they scale with the size L of the system:

$$\langle V^q\rangle \sim L^{\nu(q)} \tag{5}$$

where $\nu(q)$ gives a non-trivial family of exponents describing the growth.

For fingering in a planar geometry with a constant velocity V_0 a long way from the fluid boundary, the two sets of exponents are related as follows:

$$\nu(q)=-(q-1)D(q)-d_f+q. \tag{6}$$

Notice that the sum of the probabilities p_i is unity whilst the total flux across the fluid boundary is V_0L.

We now look at the distribution of p_i (Halsey *et al* 1986a, b). Let $n(p)\,\mathrm{d}\log(p)$ be the number of sites on the cluster which have a probability p_i of growth in the small interval $\log(p)\pm\mathrm{d}\log(p)$. We write $Z(q)$ as

$$Z(q)=\int \mathrm{d}\log(p)n(p)p^q \tag{7}$$

then assuming the scaling $p\sim L^{-\alpha}$ and $n(p)\sim L^{f(\alpha)}$ we obtain

$$Z(q)=\int L^{f(\alpha)-q\alpha}\log(L)\,\mathrm{d}\alpha. \tag{8}$$

Evaluating the integral by steepest descents we can say that

$$(q-1)D(q)=q\alpha(q)-f(q) \tag{9}$$

where $\partial f/\partial\alpha=q$. We can then plot f as a function of α and say that the contributions to the moments come from $L^{f(\alpha)}$ points with growth probabilities scaling as $L^{-\alpha}$.

We solve Laplace's equation on a simple model fractal in two dimensions to find an analytic expression for $\nu(q)$. This gives us a physical understanding of the origin of the scaling and its nature. The method can be easily extended to other systems with a fractal boundary condition.

Consider the cosine wave whose wavelength λ_1 is equal to the size of the system L:

$$y=\lambda_1\varepsilon_1\cos(k_1s_0) \qquad k_1=2\pi/\lambda_1 \tag{10}$$

as shown in figure 1(a) where s_1 is the coordinate along the curve. Now buckle the surface with another wave m_1 times smaller than the first. That is, $k_2 = m_1 k_1$ and the amplitude is $\varepsilon_1 \lambda_2$. This is not the simple addition of two waves; the second is added perpendicular to the surface of the first (figure 1(b)):

$$y_\perp = \lambda_2 \varepsilon_2 \cos(k_2 s_1 + \theta_1). \tag{11}$$

Repeat this process with a succession of waves until the shortest wavelength is equal to the pore size. We should be able to describe any connected shape in two dimensions with a suitable set of ε, m and θ. For simplicity we take ε and m to be the same at each stage and the θ to be random and independent of each other: $\varepsilon < 1$ and $m > 1$. If we have n stages of buckling then $m^n = L$ and so

$$n = \log L / \log m. \tag{12}$$

The boundary is self-similar. This process may be regarded as a smooth but statistical adaptation of the Koch curve construction (Mandelbrot 1982). It is only well defined in two dimensions, but all the equations below easily generalise to higher dimensions. Notice that we are using only O(log L) coordinates to describe an interface of length L. We do not know if this amount of information would be sufficient to describe any self-similar perimeter.

We measure d_f by considering the increase in apparent arc length at each change of scale (Mandelbrot 1982):

$$d_f = 1 + \varepsilon^2/4 \log m + O(\varepsilon^4). \tag{13}$$

Fully developed curves are shown in figure 2. This model is chosen because, when each buckling is well separated in length, we can find the velocity distribution on the curve analytically as a power series in ε. This model cannot, however, account for the highly branched shapes of DLA clusters. Nevertheless, numerical simulations (King 1987) indicate that for finite viscosity ratios, the fingering displacement is compact but the interface is fractal, with a lower dimension (i.e. less branched) than DLA, which decreases with decreasing viscosity ratio. We demonstrate the scaling structure of a

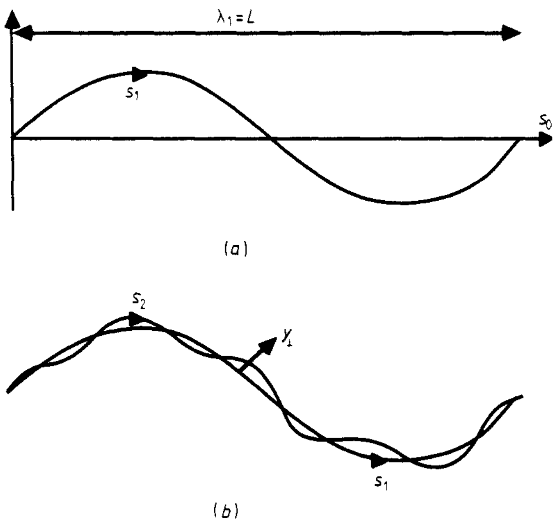

Figure 1. The first (a) and second (b) stages in the buckling of the interface.

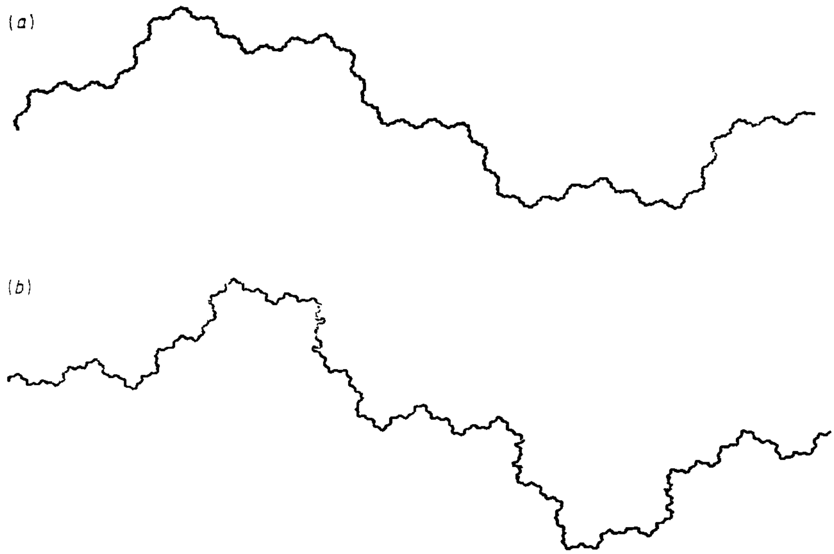

Figure 2. The interface after many many stages of buckling. (*a*) $m = 5$, $\varepsilon = 0.8$ and $d_f = 1.10$. (*b*) $m = 3$, $\varepsilon = 0.8$, $d_f = 1.15$. The large value of ε is chosen so that the discrete stages of buckling can be clearly seen. Notice particularly that curves can contain overhangs even where d_f is close to unity. When the stages of buckling are well separated in length (as in (*a*)) we are able to find, analytically, the scaling of the velocity distribution on such a curve as a power series in ε, the dimensionless amplitude of each crinkle.

Laplacian field on this fractal boundary and its dependence on the viscosity ratio and the geometry.

First consider a flat boundary perturbed by a single small cosine wave. We solve Laplace's equation as a power series in the dimensionless amplitude ε of the wave. The boundary conditions are that the pressure ϕ and the normal velocity are continuous at the interface, with a constant velocity V_0 at infinity (see (1)). This calculation is similar to the stability analysis of Chouke *et al* (1959) with no surface tension. Because we are dealing with a finite viscosity ratio, we solve for the pressure in both fluids. The velocity $V(s_0)$ normal to the boundary is

$$V(s_0) = V_0(1 + \gamma\varepsilon \cos s_0 - \tfrac{1}{2}\varepsilon^2 \sin^2 s_0) \tag{14}$$

where

$$\gamma = (\eta_{\text{oil}} - \eta_{\text{water}})/(\eta_{\text{oil}} + \eta_{\text{water}}). \tag{15}$$

The surface crinkling makes a small perturbation to the mean velocity V_0. If the oil is more viscous than the water displacing it then the flow is unstable and the initial buckling grows.

In our model, consider the fluid flow with the addition of the next largest wave. If $m \gg 1$ then we assume that a point on the smaller wave perturbs the velocity field about the local mean $V(s_0)$ and not V_0:

$$V(s_1) = V(s_0)[1 + \gamma\varepsilon \cos(s_1 + \theta_1) - \tfrac{1}{2}\varepsilon^2 \sin^2(s_1 + \theta_1)]. \tag{16}$$

Then after n stages we can write a recursion relation

$$V(s_n) = V(s_{n-1})[1 + \gamma\varepsilon \cos(s_n + \theta_n) - \tfrac{1}{2}\varepsilon^2 \sin^2(s_n + \theta_n)]. \tag{17}$$

We have the same structure on many length scales. On each scale the velocity is altered by a similar small amount so that the final field is a product of the perturbations. The scaling depends upon having similar structure over a range of lengths and also that the velocity field V is multiplicative. In particular, if the normal fluid velocity on our model fractal after $n-1$ stages of buckling is $V(s_{n-1})$, where s_{n-1} is the coordinate along the interface, then, if the next perturbation at the nth stage is on a much smaller scale, we can calculate $V(s_n)$ as the velocity field from a single stage of buckling perturbed about a local mean $V(s_{n-1})$ (17). This is the origin of the scaling. A longer discussion of the variety of fields which display a multifractal behaviour is given by Coniglio (1986). We expect to see scaling if the pressure field ϕ is invariant under the transformation $\phi \to c\phi$. The velocity field V, which is the gradient of ϕ, has the same transformation properties.

We now find the moments. If V at each stage is known as a power series in ε, then we can also obtain $\nu(q)$ as a series expansion in ε. All the coordinates are independent of each other. The tangential velocity makes no contribution at this level of approximation. We have

$$\langle V(s_n)^q \rangle = \langle V(s_{n-1})^q \rangle \langle [1 + \gamma\varepsilon \cos(s_n + \theta_n) - \tfrac{1}{2}\varepsilon^2 \sin^2(s_n + \theta_n)]^q \rangle \tag{18a}$$

$$= V_0^q \langle \exp[q(\gamma\varepsilon \cos s - \tfrac{1}{2}\varepsilon^2 \sin^2 s - \tfrac{1}{2}(\gamma\varepsilon)^2 \cos^2 s)] \rangle^n \tag{18b}$$

$$\sim L^{\nu(q)}. \tag{18c}$$

Equation (18b) contains an integral along the coordinate s. We have

$$\langle V^q \rangle = V_0^q I^n \tag{19a}$$

where, to second order,

$$I = (1 - \varepsilon^2/4)/2\pi \int_0^{2\pi} \exp[q(\gamma\varepsilon \cos s - \tfrac{1}{2}\varepsilon^2 \sin^2 s - \tfrac{1}{2}(\gamma\varepsilon)^2 \cos^2 s)] \times (1 + \tfrac{1}{2}\varepsilon^2 \sin^2 s)\, \mathrm{d}s. \tag{19b}$$

For $\gamma = 1$, I contains two standard integrals: $I = \exp(-\tfrac{1}{2}q\varepsilon^2)\, [I_0(q\varepsilon) - (\varepsilon^2/4)I_2(q\varepsilon)]$, where I_0 and I_2 are modified Bessel functions of zeroth and second order respectively. Remember that $n = \log L/\log m$ and hence $\langle V^q \rangle = V_0^q L^{\log I/\log m} \sim L^{\nu(q)}$, and so we obtain

$$\nu(q) = \{\log[I_0(q\varepsilon) - (\varepsilon^2/4)I_2(q\varepsilon)] - \tfrac{1}{2}q\varepsilon^2\}/\log m. \tag{20}$$

The limiting forms of this expression for small and large $|q\varepsilon|$ are shown in the equations below, with the substitution $\gamma = 1$.

For intermediate γ the integral can be performed easily in two limits.

(i) For $|q\varepsilon| \ll 1$ the integral in equation (19b) is expanded as a power series in $q\varepsilon$ and the integration is performed term by term:

$$\nu(q) = (\varepsilon^2/4 \log m)[q^2\gamma^2 - (1 + \gamma^2)q] + \mathrm{O}[(q\varepsilon)^3]. \tag{21}$$

If $x = \log(V/V_0)$ then x is normally distributed. The probability distribution of x, $P(x)$, is

$$P(x) = 1/(\pi\varepsilon^2\gamma^2 n)^{1/2} \exp\{-[x + n\varepsilon^2(\gamma^2 + 1)/4]^2/(n\varepsilon^2\gamma^2)\}. \tag{22}$$

(ii) For $|q\gamma\varepsilon| \gg 1$ the integral is evaluated by the method of steepest descents. For $q>0$

$$\nu(q)=q(\gamma\varepsilon-\tfrac{1}{2}\gamma^2\varepsilon^2)/\log m-\log\{q[\gamma\varepsilon-\varepsilon^2(\gamma^2-1)]\}/2\log m+\mathrm{O}[1/(q\gamma\varepsilon)] \qquad (23)$$

and for $q<0$

$$\nu(q)=-q(\gamma\varepsilon+\tfrac{1}{2}\gamma^2\varepsilon^2)/\log m-\log\{q[-\gamma\varepsilon+\varepsilon^2(\gamma^2-1)]\}/2\log m+\mathrm{O}[1/(q\gamma\varepsilon)]. \qquad (24)$$

When $\gamma=0$ there is no viscous instability and we easily derive the results:

$$\nu(q)=-q\varepsilon^2/4\log m+\mathrm{O}[(q\varepsilon^2)^2] \qquad \text{for } |q\varepsilon^2| \ll 1 \qquad (25)$$

and

$$\nu(q)=0 \qquad \text{for large } |q\varepsilon^2|. \qquad (26)$$

We use (6) to re-express $\nu(q)$ as $(q-1)D(q)$ which is plotted in figures 3 and 4. Figure 5 is a plot of f against α, in which α_∞ is given by

$$\alpha_\infty=1-(\gamma\varepsilon-\tfrac{1}{2}\gamma^2\varepsilon^2)/\log m. \qquad (27)$$

The graphs of $(q-1)D(q)$ (figures 3 and 4) have the following two features.

(i) For large q both positive and negative:

$$(q-1)D(q)=D_{\pm\infty}q \qquad q\to\pm\infty. \qquad (28)$$

The graph is linear. The high-order positive moments are dominated by the most protruding tips. If $V_{\max}$ is the maximum speed of the interface which scales as L^m, then

$$\langle V^q\rangle \sim V^q_{\max} \sim L^{mq}. \qquad (29)$$

The scaling is simple. Similarly, for large negative moments, $\langle V^q\rangle \sim V^q_{\min}$. In DLA (Halsey *et al* 1986b, Amitrano *et al* 1986) and for experiments on viscous fingering with an infinite viscosity ratio (Nittmann *et al* 1987) these asymptotic regimes are found to exist for $|q|>2$.

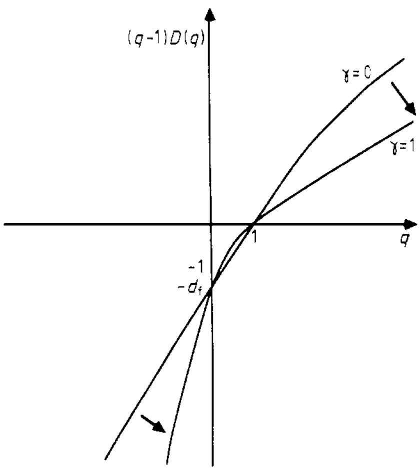

Figure 3. Schematic graph of the exponents of the growth probability distribution showing the effect of increasing the viscosity ratio γ from 0 to 1, as indicated by the arrows.

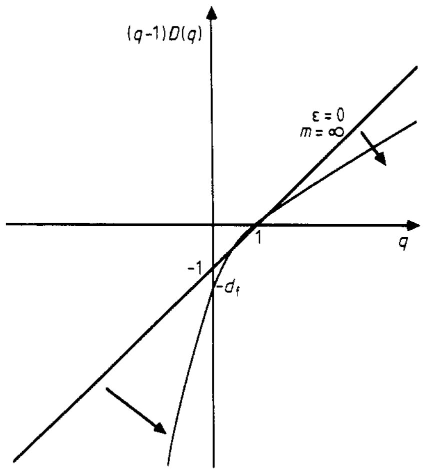

Figure 4. Schematic graph of the exponents of the growth probability distribution showing the effect of the buckling coordinates, decreasing m and increasing ε as indicated by the arrows. The dependence is similar for all viscosity ratios.

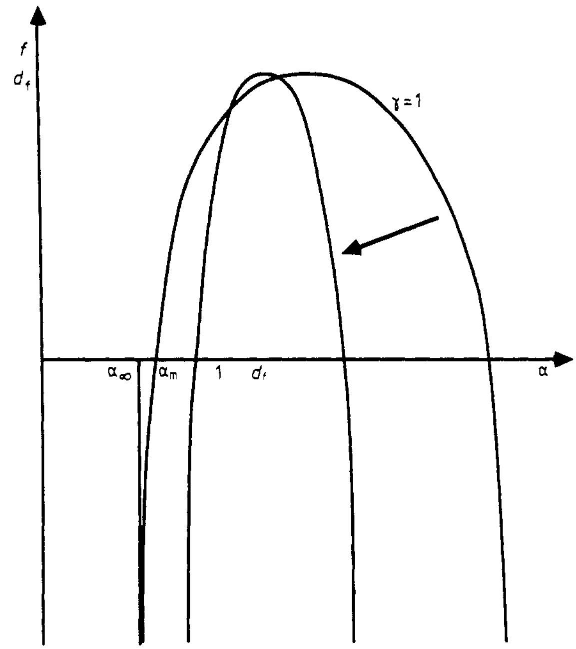

Figure 5. Fractal dimension f plotted against fractal singularity α showing the effect of decreasing the viscosity ratio γ as indicated by the arrow.

(ii) For the smaller moments many portions of the interface give significant contributions and the scaling departs from linearity. This roughly parabolic region about $q=0$ is seen numerically (Amitrano *et al* 1986) and experimentally (Nittmann *et al* 1987). The probability distribution of the velocity is extremely broad. This rich 'mutifractal' behaviour is also observed in Meakin's screened growth model (Meakin 1983, 1986, Meakin *et al* 1985) and the electrical current and flicker noise distributions through fractal resistor networks (de Arcangelis *et al* 1985, Rammal *et al* 1985).

The value of α, α_m, where $f=0$ and q is positive represents the strongest singularity seen on a typical interface, i.e. the most likely maximum tip velocity. This fastest growing tip will dominate the growth and indicates the mass dimension d_m of the displacement if we allowed the system to develop. It can be shown that (Turkevich and Sher 1985):

$$d_m = 1 + \alpha_m. \tag{30}$$

DLA is all interface, so $d_m = d_f$, but for $\gamma < 1$ the fingers will fatten, indicating that d_m and d_f are unlikely to be equal. It is possible that d_m may equal 2 (the dimension of space here) while $d_f \neq 1$ (the fluid boundary is still fractal). In our model, for given ε and m, α_m increases with decreasing γ.

The model also allows, however, for smaller values of α with negative f. This corresponds to rare, atypical arrangements of the interface and arises from the logarithmic correction to the linear form of $(q-1)D(q)$ as $q \to \pm\infty$; see (23) and (24). It is possible, but unlikely, that all our waves could be in phase; in fact the largest tip usually derives from most of the bucklings being nearly in phase. Similarly for DLA, any lattice animal could be grown, some possibly with stronger singularities on the perimeter than are usually seen, but this is very unusual. The negative f indicates that the chance of such an atypical cluster being produced is proportional to a negative power of L. The numerical results (Halsey *et al* 1986b, Amitrano *et al* 1986) indicate that $(q-1)D(q)$ is exactly linear as $q \to \infty$ and hence $d_m = 1 + \alpha_\infty$. This would be true if the probability of forming atypical singularities were lower than any negative power of L (for instance $\exp(-L)$). In any case $\alpha_\infty - \alpha_m$ is small for our model; that it might not have been could have arisen from the considerable approximations we have made. The existence of negative f is also found in the scaling of the moments of a Laplacian field near absorbing random and self-avoiding walks (Cates and Witten 1986).

We could try to fix ε and m to give a best fit to the numerical data, but there is no physical reason to do this. Also, the conditions $\varepsilon < 1$ and $m \gg 1$ means that we do not account for the highly branched shapes seen experimentally. This also means that the bulk of the fluid is initially non-fractal ($d_m = 2$). If we did allow the interface to advance, within these constraints, the self-similarity of the structure would be lost. We find that

$$\partial\varepsilon/\partial t = k_i\varepsilon + O(\varepsilon^3). \tag{31}$$

All the modes grow exponentially and the smallest ones fastest. However, in the great variety of experimental situations already mentioned, the interface does remain fractal and our work predicts a rich scaling structure in such cases. A more sophisticated approach taking into account higher-order terms in (31) might be able to place a condition on the coordinates ε_i such that $\nu(q)$ does not change with time.

All we have been able to do is solve Laplace's equation on a simple static fractal model. However, the analysis gives us an understanding of the rich and complex behaviour of a fractal boundary in an external field as well as indicating the scaling properties of large-scale viscous fingering with a finite viscosity ratio, which has yet to be studied. We can also see clearly the effect of geometry on the scaling structure. The approach could also be applied to other systems with a fractal boundary condition.

MJB is grateful for a CASE award from British Petroleum and the SERC.

References

Amitrano C, Coniglio A and di Liberto F 1986 *Phys. Rev. Lett.* **57** 1016
Ben-Jacob F, Godbey R, Goldenfield N D, Koplik J, Levine H, Mueller T and Sander L M 1985 *Phys. Rev. Lett.* **55** 1315
—— 1986 *Phys. Rev. Lett.* **57** 1903
Brady R and Ball R C 1984 *Nature* **309** 225
Buka A, Kertesz J and Vicsek T 1986 *Nature* **323** 424
Cates M E and Witten T A 1986 *Phys. Rev. Lett.* **56** 2497
Chen J-D and Wilkinson D 1985 *Phys. Rev. Lett.* **55** 1892
Chouke R L, Van Meurs P and Van der Poel C 1959 *Trans. AIME* **216** 188
Coniglio A 1986 *Statphys 16* ed H E Stanley (Amsterdam: North-Holland)
Daccord G, Nittmann J and Stanley H E 1986 *Phys. Rev. Lett.* **56** 336
de Arcangelis L, Redner S and Coniglio A 1985 *Phys. Rev.* B **31** 4725
Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986a *Phys. Rev.* A **33** 1141
Halsey T C, Meakin P and Procaccia I 1986b *Phys. Rev. Lett.* **56** 854
Grier D, Ben-Jacob E, Clarke R and Sander L M 1986 *Phys. Rev. Lett.* **56** 1260
Kaufman J H *et al* 1986 *Phys. Rev. Lett.* **56** 1932
King P R 1987 *J. Phys. A: Math. Gen.* **20** L529
Langer J S 1980 *Rev. Mod. Phys.* **52** 1
Maloy K-J, Feder J and Jossang T 1985 *Phys. Rev. Lett.* **55** 2688
Mandelbrot B B 1974 *J. Fluid Mech.* **62** 331

—— 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
Meakin P 1983 *Phys. Rev.* B **28** 6718
—— 1986 *Phys. Rev.* A **34** 790
Meakin P, Leyvraz F and Stanley H E 1985 *Phys. Rev.* A **31** 1195
Miller A, Knoll W and Mohwald H 1986 *Phys. Rev. Lett.* **56** 2633
Niemeyer L, Pietronero L and Wiessmann H J 1984 *Phys. Rev. Lett.* **52** 1033
Nittmann J 1986 *Statphys 16* ed H E Stanley (Amsterdam: North-Holland) pp 124–33
Nittmann J, Daccord G and Stanley H E 1985 *Nature* **314** 141
—— 1986 *Fractals in Physics* ed L Pietronero and E Tosatti (Amsterdam: North-Holland) pp 193–202
Nittmann J, Stanley H E, Touboul E and Daccord G 1987 *Phys. Rev. Lett.* **58** 619
Paterson L 1984 *Phys. Rev. Lett.* **52** 1625
Rammal R, Tannous C, Breton P and Tremblay A M S 1985 *Phys. Rev. Lett.* **54** 1718
Sawada Y, Dougherty A and Gollub J P 1986 *Phys. Rev. Lett.* **56** 1260
Turkevich L and Scher H 1985 *Phys. Rev. Lett.* **55** 1026
Van Damme H, Obrecht F, Levitz P, Gatineau L and Laroche C 1986 *Nature* **320** 731
Witten T and Sander L M 1981 *Phys. Rev. Lett.* **47** 1400